Substitution of Eqs. (4) into Eqs. (1) yields

$$\sigma_{11} = \sigma_{22} = \frac{E}{1-\nu}\left[\frac{\int_{-h/2}^{h/2} \frac{E\epsilon_T}{1-\nu}\,dz}{\int_{-h/2}^{h/2} \frac{E}{1-\nu}\,dz} - \epsilon_T\right] \qquad (5)$$

$$\sigma_{12} = 0$$

Equations (5) and (3) define the resultants:

$$T_1 = T_2 = T_{12} = M_{12} = M_{21} = H = S = 0$$
$$M_1 = M_2 \qquad (6)$$

The resultants (6) satisfy Eqs. (2), and hence the stresses of Eqs. (5) are the desired solution.

This analysis of closed, thin shells of constant thickness, subjected to temperature variations through the thickness, shows that, within the limits of thin-shell theory, the tangential thermal stresses are independent of the shape of the shell. The radial stresses depend upon the local curvatures and hence vary over the shell. The expressions for thermal stresses, Eqs. (5), include the effects of layers of different materials through the shell thickness as well as variations of mechanical properties with temperature.

Reference

[1] Novozhilov, V. V., *The Theory of Thin Shells* (P. Noordhoft Ltd., The Netherlands, 1959), Chap. I.

Modification of Encke's Method Suitable for Analog Solution

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Nomenclature

$\mathbf{e}_h$ = a unit vector in the plane formed by $\mathbf{r}_s$ and $\dot{\mathbf{r}}_s$ and perpendicular to $\mathbf{r}_s$
$\mathbf{F}$ = force vector
g_1, g_2 = relative gravitational components defined by Eqs. (6) and (7)
δh = difference between $|\mathbf{r}|$ and $|\mathbf{r}_s|$
m = particle mass
$\mathbf{r}$ = particle position vector
$\mathbf{r}_s$ = reference particle position vector
$\mathbf{S}_s$ = relative range vector
S_x = inertial component of $\mathbf{S}_s$ aligned perpendicular to $\mathbf{r}$ at zero time and in the direction of motion
S_y = inertial component of $\mathbf{S}_s$, perpendicular to the reference plane, and completing an orthogonal system with S_x and S_z
S_z = inertial component of $\mathbf{S}_s$ aligned along $-\mathbf{r}$ at zero time
μ = planetary gravitational constant
$\delta\rho$ = nondimensional altitude difference defined by Eq. (8)
χ = angle between $\mathbf{r}$ and $\mathbf{r}_s$
$\cdot$ = first derivative with respect to time
$\cdot\cdot$ = second derivative with respect to time

The method proposed by J. F. Encke for handling small perturbations in the vicinity of a central force field is peculiarly suitable for real-time solution of the orbital flight equations on an analog computer. The method requires that differential accelerations due to perturbations from a two-body reference orbit be integrated; thus, the kinematic solution for position becomes the relative position vector. The number of significant figures needed in the calculation is reduced to a level where computation with four digits becomes comparable to a six-digit solution achieved by integrating the total accelerations.

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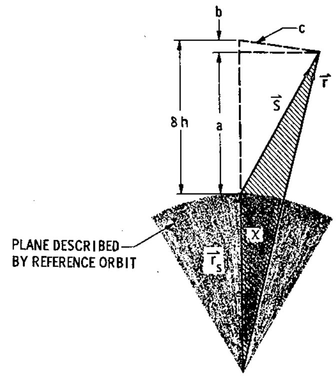
Fig. 1 Relative orientation of r and $\mathbf{r}_s$ vectors



The vector differential equation of motion in a central force field is given as

$$\ddot{\mathbf{r}} = -(\mu/r^3)\mathbf{r} + (\mathbf{F}/m) \qquad (1)$$

Now proceed to derive the differential equation of motion of the relative range vector $\mathbf{S}$ referenced to a point particle operating in freefall at a position $\mathbf{r}_s$ such that

$$\mathbf{r} = \mathbf{r}_s + \mathbf{S}_s \qquad (2)$$

From this it follows that $\dot{\mathbf{r}} = {}^{\cdot}{}_s + \dot{\mathbf{S}}_s$. Substitution of (2) into (1) yields

$$\ddot{\mathbf{S}}_s = -\ddot{\mathbf{r}}_s - (\mu/r^3)\mathbf{r}_s - (\mu/r^3)\mathbf{S}_s + (\mathbf{F}/m) \qquad (3)$$

For the conditions stated concerning $\mathbf{r}_s$,

$$\ddot{\mathbf{r}}_s = -(\mu/r_s^3)\mathbf{r}_s \qquad (4)$$

Substitution of (4) into (3) yields

$$\ddot{\mathbf{S}}_s = -\mathbf{r}_s\frac{\mu}{r_s^3}\left(\frac{r_s^3}{r^3} - 1\right) - \frac{\mu}{r_s^3}\frac{r_s^3}{r^3}\mathbf{S}_s + \frac{\mathbf{F}}{m} \qquad (5)$$

Now define

$$g_1 \equiv (\mu/r_s^3)[(r_s^3/r^3) - 1] = -g_2\delta\rho[2 + \delta\rho + (1+\delta\rho)^2] \qquad (6)$$

$$g_2 \equiv (\mu/r_s^3)(r_s^3/r^3) = (\mu/r_s^3)[1/(1+\delta\rho)^3] \qquad (7)$$

$$\delta\rho \equiv (r/r_s) - 1 \qquad (8)$$

Then the resulting relation describing $\mathbf{S}$ is obtained by twice integrating

$$\ddot{\mathbf{S}}_s = -\mathbf{r}_s g_1 - \mathbf{S}_s g_2 + (\mathbf{F}/m) \qquad (9)$$

It is evident that the subtraction described by the identity of Eq. (8) does not lead to the desired result of fewer significant

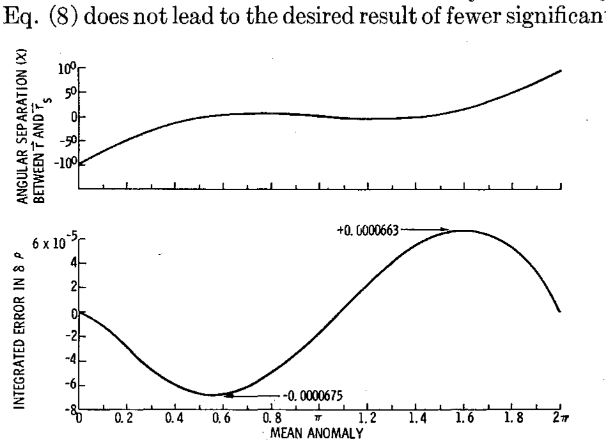


Fig. 2 Integrated error in $\delta\rho$ due to small angle approximation made for χ

Table 1 Integrated errors resulting from linearization of g_1 and g_2 functions

Change in mean anomaly from initial point	Correct value of $\delta\rho$	Error in $\delta\rho$ due to first-order approximation	Error in $\delta\rho$ due to second-order approximation	Error in $\delta\rho$ due to third-order approximation
0	−0.0008626	0	0	0
$\pi/2$	−0.0699590	−0.0041556	0.0002451	0.0000124
π	0.0112128	0.0042206	0.0016468	0.0001375
$3\pi/2$	0.0704753	0.0390562	0.0017051	0.0002118
2π	−0.0008626	...	−0.0076109	0.0002605

figures but actually would require more. The original solution to this problem proposed by Encke involves a series expansion of the perturbed position vector which yields accurate results when the terms are small. A modification[1] of this technique is suggested by the following development.

From inspection of Fig. 1, it is observed that segment a may be expressed as $\mathbf{S}_s \cdot \mathbf{r}_s / r_s$. Further inspection shows that

$$\sin\chi = (1/r)[(\mathbf{S}_s \cdot \mathbf{e}_h)^2 + S_y^2]^{1/2}$$

where $\mathbf{e}_h$ is a unit vector normal to $\mathbf{r}_s$ in the plane of motion of $\mathbf{r}_s$ and in the direction of $\dot{\mathbf{r}}_s$. It is also apparent that the chord c between $\mathbf{r}$ and $[\mathbf{r}_s + \delta h(\mathbf{r}_s/r_s)]$ is given approximately by χr. The angle subtended by this chord is $\frac{1}{2}\chi$. Therefore, the segment b is given by $(\chi r)\sin(\frac{1}{2}\chi)$. For $\chi \leq 10°$, good approximation for segment b is obtained by letting $\sin\chi = \chi$, resulting in

$$\delta h = (1/2r)[(\mathbf{S}_s \cdot \mathbf{e}_h)^2 + S_y^2] + \mathbf{S}_s \cdot (\mathbf{r}_s/r_s) \tag{10a}$$

or

$$\delta\rho = (1/2rr_s)[(\mathbf{S}_s \cdot \mathbf{e}_h)^2 + S_y^2] + (\mathbf{S}_s/r_s)\cdot(\mathbf{r}_s/r_s) \tag{10b}$$

The possibility of linearizing the expression for g_1 and g_2 when $\delta\rho$ is on the order of 0.1 is considered next. If the appropriate expansions and substitutions are performed, the differential equation for $\mathbf{S}_s$ becomes

$$\ddot{\mathbf{S}}_s \sim -\frac{\mu}{r_s^2}\left[\frac{\mathbf{S}_s}{r_s} - 3\delta\rho\left(\frac{\mathbf{r}_s}{r_s} + \frac{\mathbf{S}_s}{r_s}\right) + 6(\delta\rho)^2\frac{\mathbf{r}_s}{r_s} + 0(\delta\rho^3)\right] + \frac{\mathbf{F}}{m} \tag{11}$$

Table 1 illustrates the integrated error introduced by various orders of approximation for a perturbation from a circular reference orbit such that $\delta\rho \leq 0.071$. With reference to this table, the following conclusions regarding linearization are immediate: 1) first-order linearization will not provide sufficient accuracy for this problem; and 2) if second- or third-order terms are retained, the mechanization is only slightly less complex than using the exact formulation called for in Eqs. (6) and (7).

Figure 2 demonstrates the propagation of error in $\delta\rho$ during one orbital period due to the approximation used in developing Eqs. (10). The data presented in Table 1 and Fig. 2 were generated by digital equipment. Figure 3 illustrates the accuracy achievable with analog equipment by using this method. The trajectory shown covers one-half an orbital period, starting at perigee with $\delta\rho = -0.07048$ and finishing at apogee. The data points indicated on the curve were generated from the exact two-body solution for comparison purposes. The agreement is seen to be excellent.

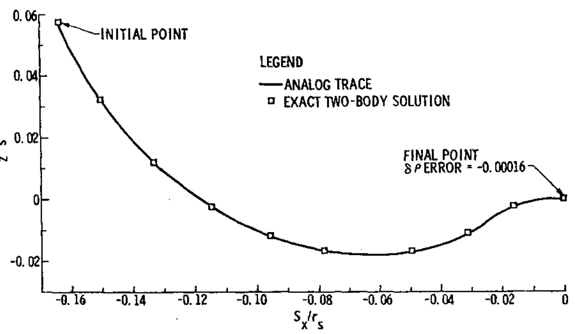


Fig. 3 Solution of Eq. (9) with analog computer

Reference

[1] Markson, E. E., "Mathematical model for lunar mission simulation," Martin Marietta Corp., Baltimore Div., Rept. ER 12697 (November 1962).

Free-Convection Viscoelastic Flow Past a Porous Flat Plate

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THE problem of shear flow of a viscoelastic fluid past a porous flat plate has been considered by Gupta.[1] In the present note an attempt has been made to generalize this problem by taking into account the effect of free-convection when a body force g per unit mass is acting in the negative x direction parallel to the plate. By considering a suitable type of stress-strain relation and assuming the fluid to be semi-incompressible, the solution has been obtained in closed form.

The general stress-strain relation for a viscoelastic fluid is given as

$$\sigma_j^i + \tau\tilde{\sigma}_j^i = 2\mu(e_j^i - \tfrac{1}{3}\Delta\delta_j^i) \tag{1}$$

where σ_j^i is the extra stress tensor (in the sense of Noll[2]), τ is the elastic constant having dimensions of time, μ is the coefficient of viscosity, Δ is dilatation, and e_j^i is the rate-of-strain tensor. The term $\tilde{\sigma}_j^i$ appearing in Eq. (1) denotes its rate of change, which, following Truesdell,[3] is taken as

$$\tilde{\sigma}_j^i = (\partial\sigma_j^i/\partial t) + \sigma^i{}_{j,k}v^k + \sigma_j^i v_{,k}^k - \sigma^{ik}v_{j,k} - \sigma_j^k v_{,k}^i \tag{2}$$

Similar to the analysis of Gupta, it is assumed that all quantities, except the pressure p, depend on y only. Thus the equations of motion and continuity governing the problem become

$$\rho v(du/dy) = -(\partial p/\partial x) + (d\sigma_y^x/dy) - \rho g \tag{3}$$

$$\rho v(dv/dy) = -(\partial p/\partial y) + [d(\lambda\Delta)/dy] + (d\sigma_y^y/dy) \tag{4}$$

and

$$d(\rho v)/dy = 0 \tag{5}$$

respectively, where λ denotes the coefficient of bulk viscosity, $\Delta = e_i^i = dv/dy$, and σ_x^x, σ_y^x, σ_y^y are the stress components satisfying Eq. (1).

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